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Fuzzy Preference Relations

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Abstract- In this paper, we study the weak, strong and indifference fuzzy relations in a 'set of alternatives', in which a semigroup structure is defined. This enables to define a fuzzy congruence relation of 'indifference' in the set and to treat the set in a better way for making exact choice

Keywords: Fuzzy preference, set of alternatives, semigroup, indifference relation

1. INTRODUCTION

In Economics we come across many choice functions, all of them are rules to get a proper choice from a set of alternatives. Since our preferences are often imprecise, it is appreciable to use fuzzy preference relations to make better choice rules.

The theory of 'fuzzy semigroup' is a part of fuzzy algebra, which is an important branch of fuzzy mathematics. The notion of a fuzzy congruence on a semigroup is studied by many authors[1,6,8,9,10,11]. The set X of alternatives can be realized as a semigroup by introducing a semigroup structure in X. We make use of the fuzzy weak preference relation R on X for this purpose. As in the case of crisp binary relations, it is possible to define a fuzzy binary relation among the elements of a SINGLE SET X.

2. PRELIMINARIES

2.1 Definition

Let X be the universal set. A fuzzy set in X is a set of ordered pairs,
$A = \{ (x, \mu_A(x)) \ ; \ x \in X \}$, where $\mu_A : X \rightarrow [0,1]$ is called the membership function of A in X and [0,1] is called the membership set[12].

2.2 Definition

Let X and Y are two crisp sets. A fuzzy binary relation R on $X \times Y$ is the set of ordered pairs $\{((x, y), \mathrm{R}(x, y)) : (x, y) \in X \times Y \}$. A fuzzy binary relation R can be conveniently represented by a matrix, called the membership matrix of R. Fuzzy binary relation from a set X to itself is usually expressed as a fuzzy binary relation on X[12].

2.3 Definition

A fuzzy binary relation R on a set X is

i. reflexive iff $\mathrm{R}(x, x) = 1$, for all $x \in X$

ii. symmetric iff $\mathrm{R}(x, y) = \mathrm{R}(y, x)$ for all $x, y \in X$; and

iii. transitive (or more specifically max-min transitive) iff

$$\mathrm{R}(x, z) \geq \max_{y \in X} \min \{\mathrm{R}(x, y), \mathrm{R}(y, z)\}$$

for all $x, y, z \in X$

2.4 Definition

A semigroup is a non-empty set with an associative binary operation defined on it[4].

An element x of a semigroup is called regular if there exists an element e of the semigroup such that $x\,e\,x = x$. If all the elements of a semigroup are regular, then the semigroup is called a regular semigroup[5]. If an element x of a semigroup is regular, an element x' of the semigroup satisfying the equations:

$$x\,x'\,x = x \text{ and } x'\,x\,x' = x'$$

is called an inverse of x. If every element of a semigroup has a unique inverse in the semigroup, we call the semigroup an inverse semigroup[2]. An element i of a semigroup is called an idempotent, if $i^2 = i$. If every element of a semigroup is an idempotent, then the semigroup is called a 'band'.

2.5 Definition

A relation R on a semigroup G is

i. left compatible with respect to the operation in G,
if for all $s, t, a \in \mathrm{G}$, $(s, t) \in \mathrm{R} \Rightarrow (a\,s, a\,t) \in \mathrm{R}$

ii. right compatible if,
for all $s, t, a \in \mathrm{G}$, $(s, t) \in \mathrm{R} \Rightarrow (s\,a, t\,a) \in \mathrm{R}$

iii. compatible if it is both left and right compatible.

2.6 Definition

A compatible equivalence relation is called a congruence [3, 7].

2.7 Definition

A fuzzy compatible similarity relation on a semigroup G is called a fuzzy congruence.

3. Semigroup of Alternatives

Let X be the set of alternatives from which a choice has to be made. Let every $x \in X$ be known and pair wise comparable. A fuzzy binary relation R is defined on X as follows:

3.1 Definition

The fuzzy weak preference relation R on X is defined as:

R(x, y) = the degree to which the alternative x is 'at least as good as' the alternative y.

Let R be a transitive fuzzy weak preference relation on the set X of alternatives. Define an operation '•' on X by :

$$x \bullet y = \begin{cases} x, & \text{if } R(x, y) \geq R(y, x) \\ y, & \text{otherwise}; \end{cases} \quad \text{for every } x, y \in X.$$

Since R is transitive, '•' is an associative, binary operation on X. Hence X is a semigroup under this operation. The binary operation '•' in fact gives an agent's (consumer's) preference on a set of alternatives.

For any $x, y \in X$, either $R(x, y) < R(y, x)$, or $R(x, y) = R(y, x)$ or $R(x, y) > R(y, x)$. We write $x \geq y$ if $R(x, y) > R(y, x)$ or $R(x, y) = R(y, x)$. Then, ' $\geq$ ' is a partial order in X. Let $e \in X$ be such that $R(x, e) \geq R(e, x)$, for every $x \in X$.

Let $y \in X$. Then, $y\,e\,y = y\,(e\,y) = y\,y = y$. Therefore, y is a regular element of X. Since y is arbitrary, every element of X is regular and hence X is a regular semigroup. For any $y \in X$, let $y' = e\,y\,e$. Then, $y\,y'\,y = y$.

But,

$$y'\,y\,y = \begin{cases} y, & \text{if } R(y, e) > R(e, y) \\ e, & \text{if } R(y, e) = R(e, y) \end{cases}$$

Hence elements of X have only partial inverses. Since each element of X is an idempotent, X is a band. Clearly X is not commutative, since, $x \bullet y \neq y \bullet x$ in general in X.

Using the fuzzy weak preference relation R, we define another fuzzy preference relation I in the set of alternatives as follows:

3.2 Definition

For any $(x, y) \in X^2$, $I(x, y) = \min\{R(x, y), R(y, x)\}$. I is called the fuzzy indifference relation on X.

3.3 Proposition

I is similarity relation on X if it is derived from a reflexive and transitive fuzzy weak preference relation.

Proof:

I is clearly symmetric. Also I is reflexive if R is so.

For each $x, y, z \in X$,

$$\min\{I(x, y), I(y, z)\} = \min\{\min[R(x, y), R(y, x)], \min[R(y, z), R(z, y)]\}$$
$$= \min\{\min[R(x, y), R(y, z)], \min[R(z, y), R(y, x)]\}$$
$$\leq \min\{[R(x, z), R(z, x)]\} = I(x, z)$$

i.e., transitivity of R ensures that of I. hence I is a similarity relation on X.

3.4 Lemma

For each $x \in X$, define a fuzzy subset I_a of X such that $I_a(x) = I(a, x)$ and let $b \in X$. Then, $I_a = I_b$ if and only if $I(a, b) = 1$.

Proof :

Let $I_a = I_b$. Then, $I(a, b) = I_a(b) = I_b(b) = I(b, b) = 1$, by the reflexivity of I. Conversely, assume that $I(a, b) = 1$. For any $x \in X$, $I_a(x) = I(a, x) \geq \min\{I(a, b), I(b, x)\}$, by the transitivity of I. i.e., $I_a(x) = \min\{1, I(b, x)\} = I(b, x) = I_b(x)$.

Hence $I_a \supseteq I_b$. By symmetry of I we also have $I_b \supseteq I_a$. $\therefore$ $I_a = I_b$.

3.5 Definition

The fuzzy subset I_a of X defined by, $I_a(x) = I(a, x)$, for every $x \in X$ is called the fuzzy similarity class of I containing $a \in X$.

3.6 Proposition

The indifference relation I defined on the semigroup X of alternatives is a fuzzy congruence relation on X.

Proof :

Let $x, y, z \in X$. Consider the following cases.

Case-1 $x \leq y \leq z$ or $y \leq x \leq z$

Then, $I(x, y) \leq 1 = I(z, z) = I(xz, yz)$.

Case-2 $z \leq x \leq y$ or $z \leq y \leq x$

Then, $I(x, y) = I(xz, yz)$.

Case-3 $x \leq z \leq y$

Then, $I(x, y) = R(x, y) \leq R(z, y) = \min\{R(z, y), R(y, z)\} = I(z, y) = I(xz, yz)$.

Case-4 $y \leq z \leq x$

Then, $I(x, y) = R(y, x) \leq R(z, x)$

$\therefore$ I(x, y) $\leq$ min{ R(z, x), R(x, z) }$\leq$ I(x, z) $\leq$ I($x z, y z$).
Thus for any $x, y, z \in X$, I(x, y) $\leq$ I($x z, y z$).
Hence I is fuzzy right compatible. Similarly we can prove that I is fuzzy left compatible. As I is a similarity relation on X, this proves that I is a congruence relation on X.

Since I is a fuzzy congruence relation on X, the fuzzy subset I_a of X is a fuzzy congruence class if I is containing $a \in X$.

4. CONCLUSION

Since the ralation I is a fuzzy congruence relation on X, the fuzzy subset I_a of X is a fuzzy congruence class of I containing $a \in X$. Hence we can define a quotient semigroup of X induced by I. This idea may lead us to new strategies of making choice sets from the set X of alternatives.

REFERENCES

[1] Dib, K. A and Galhum, N , 'Fuzzy Ideals and Fuzzy bi-ideals in Fuzzy Semigroups', *Fuzzy Sets and Systems* , 92 , (1997) , 103-111.

[2] George, M. , 'Certain Fundamental Congruence on a Regular Semigroup', M. Phil Dissertation, University of Kerala, Trivandrum, 1994.

[3] Gupta, K. C. and Singh, T. P. , 'Fuzzy G-Congruence on a Groupoid', *Fuzzy Sets and Systems* , 93 , (1998) , 211-222.

[4] Hammond, P. J. , 'Interpersonal Comparison of Utility : Why and How They are and Should be Made', in 'Interpersonal Comparison of Well-being',(ed) Jon Elster and John E. Roemer , 1991, Cambridge University Press.

[5] Howie, J. M. , 'An Introduction to Semigroup Theory' , Academic Press, New York , 1976.

[6] Kakwani, N. K, 'On a Class of Poverty Measures', *Econometrica* , 48, (1980) , 437-446.

[7] Kim, J.P. and Bae, D.R., 'Fuzzy Congruence in Groups', *Fuzzy Sets and Systems* , 85 , (1997) , 115-120.

[8] …….. , 'Fuzzy Congruence on Inverse Semigroups', *Fuzzy Sets and Systems* , 87 , (1997) , 335-340.

[9] McLean, R. G. , and Kummer, H. , 'Fuzzy Ideals in Semigroups' *Fuzzy Sets and Systems* , 48 , (1992) , 137-140.

[10] Pratyananda, D. , 'Fuzzy Regular and Inverse Semigroups', *Fuzzy Sets and Systems* , 91 , (1997) , 99-105.

[11] ------ , 'Lattice of Fuzzy Congruences in Inverse Semigroups', *Fuzzy Sets and Systems* , 91 , (1997) , 399-408.

[12] Zimmermann, H. J. , 'Fuzzy Set Theory- and its Applications', Allied , New Delhi , 1996.